# Dimensional analysis

## What the devil is dimensional analysis?

Dimensional analysis is elementary, my dear Watson. It's a method for determining whether an expression represents a length, an area, a volume or something else.

## Eh? Show me an example

It's probably easiest to understand dimensions and how to analyse them by going through an exam question.

**Question** In each of the following expressions, $x$, $y$, and $z$ represent lengths.
State whether each expression represents length, area, volume or none of these.

(a) $x + y$ (b) $xy$ (c) $\dfrac{xy}{z}$ (d) $\dfrac{x}{y + z}$

**Part (a)** is $x + y$. Since $x$ and $y$ are lengths, $x + y$ is a (length + length), which is a length. When you add or subtract two quantities with the same dimensions, the resulting quantity (the answer) always has the same dimensions, too.

**Part (b)** is $xy$. Since $x$ and $y$ are lengths, $xy$ is a (length × length), which is an area. When you multiply two quantities, their dimensions are multiplied too.

**Part (c)** is $\dfrac{xy}{z}$. Since $x$, $y$ and $z$ are lengths, $\dfrac{xy}{z}$ is a $\dfrac{\text{length} \times \text{length}}{\text{length}}$, which cancels down to a length. When you divide a quantity by another, the dimensions of the answer are the dimensions of the first quantity divided by the dimensions of the second quantity

**Part (d)** is $\dfrac{x}{y+z}$. Since $x$, $y$ and $z$ are lengths, $\dfrac{x}{y+z}$ is a $\dfrac{\text{length}}{\text{length} + \text{length}}$.

Because a (length + length) is a length, $\dfrac{\text{length}}{\text{length} + \text{length}}$ is a $\dfrac{\text{length}}{\text{length}}$.

The lengths in $\dfrac{\text{length}}{\text{length}}$ cancel each other out, leaving 1, or in other words, no dimensions. So the expression in part (d) is **dimensionless**.

## What's the difference between dimensions and units?

A particular dimension can be expressed in many different units.

For instance, length is a dimension and it can be expressed in many, many units: millimetres, centimetres, metres, kilometres, inches, feet, yards, miles, furlongs, nautical miles, cubits, light years, parsecs, ångströms and so on.

One dimension can be expressed in many different units but a particular unit always expresses a particular dimension. Centimetres, for example, always express length, whereas kilograms always express mass.

## Are all dimensions just lengths multiplied together?

There are three basic kinds of dimension in GCSE maths: length, time, and mass. We can use a shorthand notation to save writing out the phrase "dimensions of length" or similar. For length, write [L]. For time, write [T]. For mass, write [M].

All the quantities you will come across in maths have dimensions that are simple combinations of these base dimensions. For instance, average speed = distance travelled ÷ time taken, so its dimensions are [L] ÷ [T], because distance is a length, [L], and time taken is a time, [L].

| Type of quantity | Dimensions (long form) | Dimensions (short form) |
|---|---|---|
| Length | $\left[L\right]$ | $\left[L\right]$ |
| Area | $\left[L\right]\times\left[L\right]$ | $\left[L\right]^2$ |
| Volume | $\left[L\right]\times\left[L\right]\times\left[L\right]$ | $\left[L\right]^3$ |
| Time | $\left[T\right]$ | $\left[T\right]$ |
| Speed | $\left[L\right]\div\left[T\right]$ | $\left[L\right]\left[T\right]^{-1}$ |
| Mass | $\left[M\right]$ | $\left[M\right]$ |
| Density | $\dfrac{\left[M\right]}{\left[L\right]\times\left[L\right]\times\left[L\right]}$ | $\left[M\right]\left[L\right]^{-3}$ |

## One, two, three and more dimensions

We say that all quantities that have dimensions of length, [L], are **one-dimensional** or **1D**. Likewise, all quantities that have dimensions of of $[L]^2$ are **two-dimensional** or **2D** and those with dimensions of $[L]^3$ are **three-dimensional** or **3D**.

# Dimensions of circumference

$$c = \pi d$$

$$\text{circumference} = \text{pi} \times \text{diameter}$$

Diameter is clearly a length – it is a straight line.  So what about circumference?  We'll use dimensional analysis to work it out.

Both sides of the equation must have equal dimensions.  Pi has no dimensions – we say it is dimensionless – so multiplying diameter by pi does not change its dimensions, it is still a length.   The right hand side of the equation has dimensions of length, [L].  Therefore the left hand side (circumference) must also have dimensions of length.

# Dimensions of area

$$A_{\text{circle}} = \pi r^2$$

$$\text{area of circle} = \text{pi} \times (\text{radius squared})$$

Radius is a length, [L], so radius squared is a length squared, $[L]^2$.  Pi is dimensionless, so $\pi r^2$ also has dimensions of length squared, $[L]^2$.  This means that A, the area of a circle, must also have dimensions of $[L]^2$.  In fact, any area must have dimensions of $[L]^2$.

# Dimensions of volume

$$V_{\text{cylinder}} = \pi r^2 h$$

$$\text{volume of cylinder} = \text{pi} \times (\text{radius squared}) \times \text{height}$$

In the example above, we established that $\pi r^2$ has dimensions of length squared, $[L]^2$. The height of a cylinder, $h$, is a straight line and so has dimensions of length, [L].  The right hand side of the equation is $\pi r^2 h$ and has dimensions of $[L]^2 \times [L]$, that is $[L]^3$, so the left hand side, volume, must also have dimensions of $[L]^3$.

# Dimensions of speed

$$s = \frac{d}{t}$$

$$\text{average speed} = \text{distance travelled} \div \text{time taken}$$

Speed is the left hand side (LHS) of the equation above, and the dimensions of the LHS are always equal to the dimensions of the RHS, which are [L] ÷ [T], also written as $[L][T]^{-1}$. In other words, speed has dimensions of length ÷ time.

# Dimensions of density

$$\rho = \frac{m}{V}$$

$$\text{density of object} = \text{mass of object} \div \text{volume of object}$$

Mass has dimensions of [M] and volume has dimensions of $[L]^3$ so density had dimensions of $[M] \div [L]^3$, which is often also written as $[M][L]^{-3}$.  The symbol for density is $\rho$, the Greek letter rho.