Completing the square

What's completing the square?

Completing the square is the process of converting a normal quadratic expression (e.g. $x^2+6x+10$) into an expression in *completed square form* (e.g. $(x+3)^2+1$).

$$x^2+10x+6 \underset{\text{expanding}}{\overset{\text{completing the square}}{\rightleftarrows}} (x+3)^2+1$$

If the coefficient of x^2 is 1, use this method

0. Write down your quadratic expression.	$x^2+10x+6$
1. Set it equal to $(x+\square)^2+\square$.	$x^2+10x+6=(x+\square)^2+\square$
2. Find the coefficient of *x* and halve it. Put this in the first box.	$10 \div 2 = 5$ $x^2+10x+6=(x+\boxed{5})^2+\square$
3. Square the number in the first box. Find out what you have to add to it to give the coefficient in the quadratic expression. In this example, it's a case of finding a number that will replace the question mark: $5^2+?=6$ Put the number that works in the second box.	$10 \div 2 = 5$ $x^2+10x+6=(x+\boxed{5})^2+\boxed{-19}$ $5^2=25$ $25-19=6$

The purpose of all this working is to find an expression of the form $\left(x+\Box\right)^2+\Box$ that is equal to $x^2+10x+6$. We've done that, so we can write

$$x^2+10x+6=\left(x+5\right)^2-19$$

If the coefficient of x^2 is not 1, use this method

0. Write down your quadratic expression.	$4x^2-4x+3$
1. Set it equal to $\left(\Box x+\Box\right)^2+\Box$.	$4x^2-4x+3=\left(\Box x+\Box\right)^2+\Box$
2. Find the coefficient of x^2 and square root it. Put this in the first box.	$\sqrt{4}=2$ $4x^2-4x+3=\left(\boxed{2}x+\Box\right)^2+\Box$
3. Find the coefficient of x and divide it by double the first box. This goes in the second box.	$\sqrt{4}=2$ $4x^2-4x+3=\left(\boxed{2}x+\boxed{-1}\right)^2+\Box$ $\frac{-4}{2\times 2}=-1$
4. Square the number in the second box. Find out what you have to add to it to give the coefficient in the quadratic expression. In this example, it's a case of finding a number that will replace the question mark: $\left(-1\right)^2+?=3$ Put the number that works in the second box.	$\left(-1\right)^2=1$ $1+2=3$ $\sqrt{4}=2$ $4x^2-4x+3=\left(\boxed{2}x+\boxed{-1}\right)^2+\boxed{2}$ $\frac{-4}{2\times 2}=-1$

We have found that

$$4x^2 - 4x + 3 = (2x - 1)^2 + 2$$

Another method

Another method that is very similar to the one above involves comparing your quadratic expression to the general formula for expanding a completed square:

$$p^2x^2 + 2pqx + q^2 = (px + q)^2$$

If we compare the coefficients of x^2 and x in this general formula with those in the question, we can determine p and q.

$$4x^2 - 4x + 3 = (\square x + \square)^2 + \square$$

$$p^2x^2 + 2pqx + q^2 + r = (px + q)^2 + r$$

From this comparison, we can write equations to solve:

$p^2 = 4$ therefore $p = \sqrt{4}$ so $p = 2$	$2pq = -4$ therefore $q = \frac{-4}{2p}$ thus $q = \frac{-4}{2 \times 2}$ so $q = -1$	$q^2 + r = 3$ therefore $(-1)^2 + r = 3$ so $1 + r = 3$ which means $r = 2$

We have found the values of p, q and r, so we can write:

$$4x^2 - 4x + 3 = (px + q)^2 + r = (2x - 1)^2 + 2$$

I can't remember methods. Is there a formula?

There is a formula, but it might be harder to remember than the method! Here it is if you want it:

$$ax^2 + bx + c = \left(\sqrt{a}x^2 + \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a} + c$$

Why should I bother completing the square?

Some exam questions simply ask you to complete the square to show that you can. But more often that not, you complete the square in order to do one of two things:

1. Solve the quadratic expression
2. Plot a graph of the quadratic expression

1. Solving quadratic equations by completing the square

If you want to solve a quadratic equation like $x^2+10+6=0$, completing the square can help you do it.

$$x^2+5x+6 \xrightarrow{\text{completing the square}} (x+2.5)^2-0.25$$

Because x^2+5+6 and $(x+2.5)^2+0.25$ are always equal, we can simply replace x^2+5+6 with $(x+2.5)^2+0.25$ in the equation $x^2+5+6=0$.

$$x^2+5x+6=0 \xrightarrow{\text{convert to completed square}} (x+2.5)^2-0.25=0$$

The completed square form may be easier to solve – you just rearrange:

$$(x+2.5)^2-0.25=0$$

$$(x+2.5)^2=0.25$$

$$x+2.5=\pm\sqrt{0.25}$$

$x+2.5=0.5$ or $x+2.5=-0.5$

$x = -2$ or $x = -3$

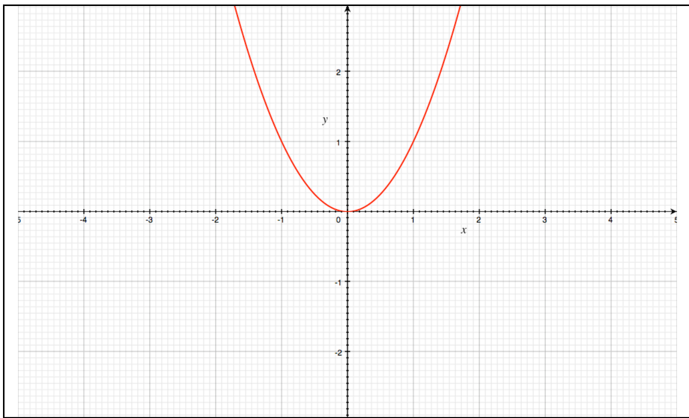
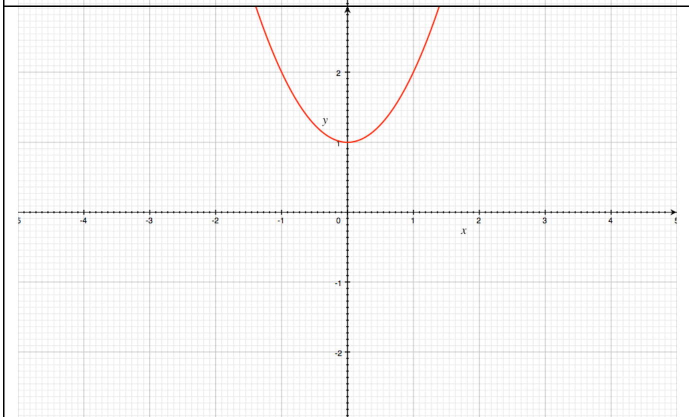
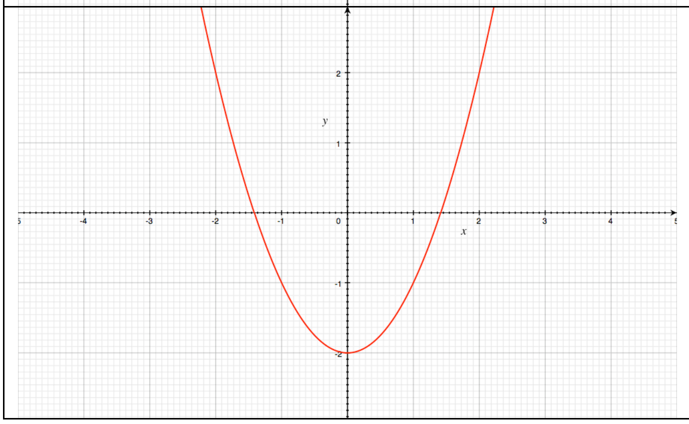
2. Plotting the graphs of quadratic equations

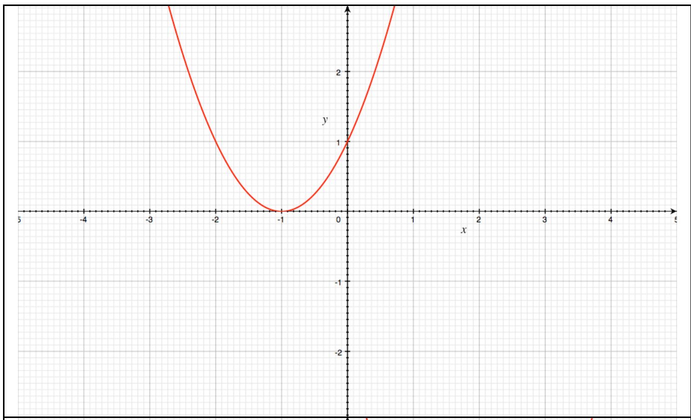
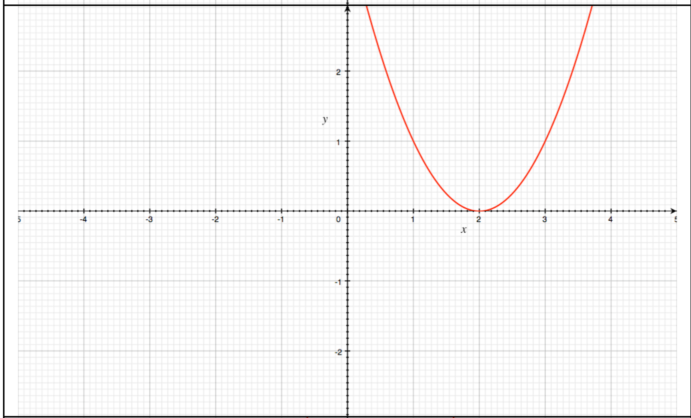
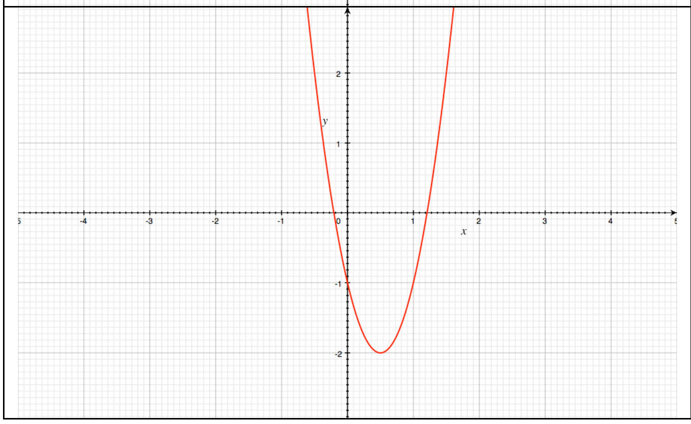
If you want to plot the graph of a quadratic equation like $y = x^2 + 10 + 6$, you have two options:

- draw up a table of values and plot them as coordinates
- complete the square and use the rules of transformations of graphs

There is no other reliable way of plotting the graph of a quadratic equation – this makes completing the square *very* useful!

The trick is to consider all quadratic graphs as transformations of the graph of $y = x^2$. All quadratics will have graphs that are roughly the same shape as this one, but may be stretched or translated or both.

	$y = x^2$ The parent of all quadratic graphs
	$y = x^2 + 1$ Same as parent but translated in the positive y direction (upwards) by 1
	$y = x^2 - 2$ Same as parent but translated in the negative y direction (downwards) by 2

	$y=(x+1)^2$ Same as parent but translated in the negative x direction (left) by 1
	$y=(x-2)^2$ Same as parent but translated in the positive x direction (right) by 2
	$y=(2x-1)^2-3$ Same as parent but translated in the positive x direction (right) by 0.5 (1÷2=0.5) , enlarged in the positive y direction by a scale factor of 4 (2^2=4) and translated in the negative y direction (down) by 3.

The importance of completing the square is that it converts quadratic equations to the form

$$y=(px+q)^2+r\ .$$

It is only once they are in this form that their graphs can be easily sketched. All that is required is to transform the graph of y = x^2 in the following way:

- stretch the graph vertically by p^2
- translate the graph to the left by $\frac{q}{p}$
- translate the graph upwards by r