Theoretical 2007
1 (c) (ii)

What the question is about
The question states that $\psi_2(x)$ and $\psi_3(x)$ are eigenfunctions of the Hamiltonian operator, and the corresponding eigenvalues are $\frac{4h^2}{8mL^2}$ and $\frac{9h^2}{8mL^2}$.

This means that:

$$\hat{H}\psi_2(x) = E_2\psi_2(x)\text{, where } E_2 = \frac{4h^2}{8mL^2}$$

and

$$\hat{H}\psi_3(x) = E_3\psi_3(x)\text{, where } E_3 = \frac{9h^2}{8mL^2}.$$

The question wants you to apply the Hamiltonian operator to $\Psi(x)$. Here's what happens:

$$\hat{H}\Psi(x) = \hat{H}\left(\psi_2(x) + 2\psi_3(x)\right)$$

$$= \hat{H}\psi_2(x) + 2\hat{H}\psi_3(x)$$

$$= E_2\psi_2(x) + 2E_3\psi_3(x)$$

$$= \frac{4h^2}{8mL^2}\psi_2(x) + 2\frac{9h^2}{8mL^2}\psi_3(x)$$

$$= \frac{4h^2}{8mL^2}\psi_2(x) + \frac{18h^2}{8mL^2}\psi_3(x)$$

$$= \frac{h^2}{8mL^2}\left(4\psi_2(x) + 18\psi_3(x)\right)$$

$$= \frac{2h^2}{8mL^2}\left(2\psi_2(x) + 9\psi_3(x)\right)$$