Theoretical 2007
1 (c) (i)

What the question is about
Normalising a wavefunction Ψ means multiplying it by a constant N so that when you integrate $N^2\Psi\Psi^*$ over all space (in 1D, this means $-\infty < x < \infty$), you get 1:

$$N^2\int_{-\infty}^{\infty}\Psi(x)\,\Psi^*(x)\,\mathrm{d}x = 1$$

You need this to be true because the value of the wavefunction at x times its complex conjugate (the complex version of squaring the wavefunction) is equal to the probability of finding the electron at x. The total probability of finding the electron must be 1 because it is certain that the electron is somewhere.

In second-year theoretical exams, they always give you a real wavefunction: no imaginary parts, so $\Psi(x) = \Psi^*(x)$ and therefore $\Psi(x)\,\Psi^*(x) = \Psi^2(x)$.

So for us the normalisation condition simplifies to:

$$N^2\int_{-\infty}^{\infty}\Psi^2(x)\,\mathrm{d}x = 1$$

The general approach
The bit they are examining you on in "normalise this wavefunction"-type questions is your ability to find N. This is the general procedure:

1. Rearrange $N^2\int_{-\infty}^{\infty}\Psi^2(x)\,\mathrm{d}x = 1$ to get $N = \dfrac{1}{\sqrt{\int_{-\infty}^{\infty}\Psi^2(x)\,\mathrm{d}x}}$.

2. Evaluate $\int_{-\infty}^{\infty}\Psi^2(x)\,\mathrm{d}x$ (i.e. do the integration, or sketch a graph, or realise its something simple you don't need to calculate).

3. Take your answer from 2 (it should be a number) and stick it into the equation from 1, $N = \dfrac{1}{\sqrt{\int_{-\infty}^{\infty}\Psi^2(x)\,\mathrm{d}x}}$. This will give you N.

4. You can now write down your normalised wavefunction: it's simply $N\Psi(x)$, but of course you have to give the value of N.

How to do this particular question
We're dealing with a particle in a box, so we know the particle is definitely in the region $0 \le x \le L$. This simplifies the integrals - we don't have to integrate over all space.

The question states that $\psi_2(x)$ and $\psi_3(x)$ are already normalised, i.e.

$\int_0^L \psi_2^2(x)\ \mathrm{d}x = 1$ and $\int_0^L \psi_3^2(x)\ \mathrm{d}x = 1$

This fact does most of the integration for you!

$$\Psi(x) = \psi_2(x) + 2\psi_3(x)$$

$$\Psi^2(x) = \left(\psi_2(x) + 2\psi_3(x)\right)^2 = \psi_2^2(x) + 4\psi_2(x)\psi_3(x) + 4\psi_3^2(x)$$

$$\int_0^L \Psi^2(x)\ \mathrm{d}x = \int_0^L \psi_2^2(x) + 4\psi_2(x)\psi_3(x) + 4\psi_3^2(x)\ \mathrm{d}x$$

$$= \int_0^L \psi_2^2(x)\ \mathrm{d}x + 4\int_0^L \psi_2(x)\psi_3(x)\ \mathrm{d}x + 4\int_0^L \psi_3^2(x)\ \mathrm{d}x$$

$$= \int_0^L \psi_2^2(x)\ \mathrm{d}x + 4\int_0^L \psi_2(x)\psi_3(x)\ \mathrm{d}x + 4\int_0^L \psi_3^2(x)\ \mathrm{d}x$$

Using the fact that $\psi_2(x)$ and $\psi_3(x)$ are normalised gives

$$= 1 + 4\int_0^L \psi_2(x)\psi_3(x)\ \mathrm{d}x + 4$$

$$= 5 + 4\int_0^L \psi_2(x)\psi_3(x)\ \mathrm{d}x$$

Wavefunctions corresponding to different states are orthogonal – $\psi_2(x)$ and $\psi_3(x)$ are different states so they should be orthogonal: $\int_0^L \psi_2(x)\psi_3(x)\ \mathrm{d}x = 0$.

Therefore $\int_0^L \Psi^2(x)\ \mathrm{d}x = 5$.

Now we can do part 3:

$N = \dfrac{1}{\sqrt{\int_0^L \Psi^2(x)\ \mathrm{d}x}}$ and $\int_0^L \Psi^2(x)\ \mathrm{d}x = 5$ (from above) so $N = \dfrac{1}{\sqrt{5}}$.

Part 4 is now easy:

$$\Psi(x) = \psi_2(x) + 2\psi_3(x)$$

$$N\Psi(x) = \frac{1}{\sqrt{5}}\Psi(x) = \frac{1}{\sqrt{5}}\psi_2(x) + \frac{2}{\sqrt{5}}\psi_3(x)\,.$$